

Asymptotic behavior for quadratic variations of non-Gaussian multiparameter Hermite random fields

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Abstract

Let $(Z_t^{q,\mathbf{H}})_{t \in [0,1]^d}$ denote a d -parameter Hermite random field of order $q \geq 1$ and self-similarity parameter $\mathbf{H} = (H_1, \dots, H_d) \in (\frac{1}{2}, 1)^d$. This process is \mathbf{H} -self-similar, has stationary increments and exhibits long-range dependence. Particular examples include fractional Brownian motion ($q = 1, d = 1$), fractional Brownian sheet ($q = 1, d \geq 2$), Rosenblatt process ($q = 2, d = 1$) as well as Rosenblatt sheet ($q = 2, d \geq 2$). For any $q \geq 2, d \geq 1$ and $\mathbf{H} \in (\frac{1}{2}, 1)^d$ we show in this paper that a proper renormalization of the quadratic variation of $Z^{q,\mathbf{H}}$ converges in $L^2(\Omega)$ to a standard d -parameter Rosenblatt random variable with self-similarity index $\mathbf{H}'' = 1 + (2\mathbf{H} - 2)/q$.

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1 Motivation and main results

In recent years, analysing the asymptotic behaviour of power variations of self-similar stochastic processes has attracted a lot of attention. This is because they play an important role in various aspects, both in probability and statistics. As far as quadratic variations are concerned, a classical application is to use them for the construction of efficient estimators for the self-similarity parameter (see e.g. [2, 15]). For a less conventional application, let us also mention the recent reference [5], in which the

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authors have used weighted power variations of fractional Brownian motion to compute exact rates of convergence of some approximating schemes associated to one-dimensional fractional stochastic differential equations.

In this paper, we deal with the quadratic variation in the context of *multiparameter Hermite random fields*. To be more specific, let $Z^{q,\mathbf{H}} = (Z_{\mathbf{t}}^{q,\mathbf{H}})_{\mathbf{t} \in [0,1]^d}$ stand for the d -parameter Hermite random field of order $q \geq 1$ and self-similarity parameter $\mathbf{H} = (H_1, \dots, H_d) \in (\frac{1}{2}, 1)^d$ (see Definition 2.1 for the precise meaning), and consider a renormalized version of its quadratic variation, namely

$$V_{\mathbf{N}} := \frac{1}{\mathbf{N}} \sum_{\mathbf{i}=0}^{\mathbf{N}-1} \left[\mathbf{N}^{2\mathbf{H}} \left(\Delta Z_{[\frac{\mathbf{i}}{\mathbf{N}}, \frac{\mathbf{i}+1}{\mathbf{N}}]}^{q,\mathbf{H}} \right)^2 - 1 \right], \quad (1.1)$$

where $\Delta Z_{[\mathbf{s}, \mathbf{t}]}^{q,\mathbf{H}}$ is the increments of $Z^{q,\mathbf{H}}$ defined as

$$\Delta Z_{[\mathbf{s}, \mathbf{t}]}^{q,\mathbf{H}} = \sum_{\mathbf{r} \in \{0,1\}^d} (-1)^{d-\sum_i r_i} Z_{\mathbf{s}+\mathbf{r}, (\mathbf{t}-\mathbf{s})}^{q,\mathbf{H}}, \quad (1.2)$$

and where the bold notation is systematically used in presence of multi-indices (we refer to Section 2 for precise definitions). As illustrating examples, observe that (1.2) reduces to $\Delta Z_{[s,t]}^{q,H} = Z_t^{q,H} - Z_s^{q,H}$ when $d = 1$, and to $\Delta Z_{[s_1, t_1], [s_2, t_2]}^{q,H_1, H_2} = Z_{t_1, t_2}^{q,H_1, H_2} - Z_{t_1, s_2}^{q,H_1, H_2} - Z_{s_1, t_2}^{q,H_1, H_2} + Z_{s_1, s_2}^{q,H_1, H_2}$ when $d = 2$.

It is well-known that each Hermite random field $Z^{q,\mathbf{H}}$ is \mathbf{H} -self-similar (that is, $(Z_{\mathbf{a}\mathbf{t}}^{q,\mathbf{H}})_{\mathbf{t} \in \mathbb{R}^d} \stackrel{(d)}{=} (\mathbf{a}^{\mathbf{H}} Z_{\mathbf{t}}^{q,\mathbf{H}})_{\mathbf{t} \in \mathbb{R}^d}$ for any $\mathbf{a} > 0$), has stationary increments (that is $(\Delta Z_{[\mathbf{0}, \mathbf{t}]}^{q,\mathbf{H}})_{\mathbf{t} \in \mathbb{R}^d} \stackrel{(d)}{=} (\Delta Z_{[\mathbf{h}, \mathbf{h}+\mathbf{t}]}^{q,\mathbf{H}})_{\mathbf{t} \in \mathbb{R}^d}$ for all $\mathbf{h} \in \mathbb{R}^d$) and exhibits long-range dependence. Also, when $q = 1$, observe that $Z^{1,\mathbf{H}}$ is either the fractional Brownian motion (if $d = 1$) or the fractional Brownian sheet (if $d \geq 2$); in particular, among all the Hermite random fields $Z^{q,\mathbf{H}}$, it is the only one to be Gaussian. When $q = 2$, we use the usual terminologies Rosenblatt process (if $d = 1$) or Rosenblatt sheet (if $d \geq 2$).

Before describing our results, let us give a brief overview of the current state of the art. Firstly, let us consider the case $q = d = 1$, that is, the case where $Z^{1,H} = B^H$ is a fractional Brownian motion with Hurst parameter H . The behavior of the quadratic variation of B^H is well-known since the eighties, and dates back to the seminal works of Breuer and Major [1], Dobrushin and Major [3], Giraitis and Surgailis [4] or Taqqu [13]. We have, as $N \rightarrow \infty$:

- If $H < 3/4$, then

$$N^{-1/2} \sum_{j=1}^N \left(N^{2H} \left(B_{j/N}^H - B_{(j-1)/N}^H \right)^2 - 1 \right) \xrightarrow{(d)} \mathcal{N}(0, \sigma_H^2).$$

- If $H = 3/4$, then

$$(N \log N)^{-1/2} \sum_{j=1}^N \left(N^{3/2} \left(B_{j/N}^H - B_{(j-1)/N}^H \right)^2 - 1 \right) \xrightarrow{(d)} \mathcal{N}(0, \sigma_{3/4}^2).$$

- If $H > 3/4$, then

$$N^{1-2H} \sum_{j=1}^N \left(N^{2H} \left(B_{j/N}^H - B_{(j-1)/N}^H \right)^2 - 1 \right) \xrightarrow{L^2(\Omega)} \text{“Rosenblatt r.v.”},$$

where “Rosenblatt r.v.” denotes the random variable which is the value at time 1 of the Rosenblatt process.

Secondly, assume now that $q = 1$ and $d = 2$, that is, consider the case where $Z^{1, \mathbf{H}}$ is this time a two-parameter fractional Brownian sheet with Hurst parameter $\mathbf{H} = (H_1, H_2)$. According to Réveillac, Stauch and Tudor [12] and with $\varphi(N, \mathbf{H})$ a suitable scaling factor, the quadratic variation of $Z^{1, \mathbf{H}}$ behaves as follows, as $N \rightarrow \infty$:

- If $\mathbf{H} \notin (3/4, 1)^2$, then

$$\varphi(N, \mathbf{H}) \sum_{i=1}^N \sum_{j=1}^N \left(N^{2H_1+2H_2} \left(\Delta Z_{[\frac{i-1}{N}, \frac{j}{N}]}^{1, \mathbf{H}} \right)^2 - 1 \right) \xrightarrow{(d)} \mathcal{N}(0, \sigma_{\mathbf{H}}^2).$$

- If $\mathbf{H} \in (3/4, 1)^2$, then

$$\begin{aligned} \varphi(N, \mathbf{H}) \sum_{i=1}^N \sum_{j=1}^N \left(N^{2H_1+2H_2} \left(\Delta Z_{[\frac{i-1}{N}, \frac{j}{N}]}^{1, \mathbf{H}} \right)^2 - 1 \right) \\ \xrightarrow{L^2(\Omega)} \text{“two-parameter Rosenblatt r.v.”}, \end{aligned}$$

where “two-parameter Rosenblatt r.v.” means the value at point $\mathbf{1} = (1, 1)$ of the two-parameter Rosenblatt sheet.

Here, we observe the following interesting phenomenon: the limit law in the mixture case (that is, when $H_1 \leq 3/4$ and $H_2 > 3/4$) is Gaussian. For the simplicity of exposition, above we have only described what happens when $d = 2$. But the asymptotic behaviour for the quadratic variation of $Z^{1,\mathbf{H}}$ is actually known for any value of the dimension $d \geq 2$, and we refer to Pakkanen and Réveillac [9, 10, 11] for precise statements.

Let us finally review the existing literature about the quadratic variation of $Z^{q,\mathbf{H}}$ in the *non*-Gaussian case, that is, when $q \geq 2$. It is certainly since it is a more difficult case to deal with that only the case where $d = 1$ has been studied so far. Chronopoulou, Tudor and Viens have shown in [2] (see also [15, 14]) that, properly renormalized, the quadratic variation of $Z^{q,H}$ converges in $L^2(\Omega)$, for *any* $q \geq 2$ and *any* value of $H \in (1/2, 1)$, to the Rosenblatt random variable. A consequence of this finding is that fractional Brownian motion is the only Hermite process ($d = 1$) for which there exists a range of parameter such that its quadratic variation exhibits *normal* convergence; indeed, for all the other Hermite processes, [2] shows that we have the convergence towards a non-Gaussian random variable belonging to the second Wiener chaos.

In the present paper, we study what happens in the remaining cases, that is, when q and d are both bigger or equal than 2. Thanks to our main result, Theorem 1.1, we now have a complete picture for the asymptotic behaviour of the quadratic variation of *any* Hermite random field.

Theorem 1.1. *Fix $q \geq 2$, $d \geq 1$ and $\mathbf{H} \in (\frac{1}{2}, 1)^d$. Let $Z^{q,\mathbf{H}}$ be a d -parameter Hermite random field of order q with self-similarity parameter \mathbf{H} (see Definition 2.1). Then $c_{1,\mathbf{H}}^{-1/2} 2^{-(d-1)/2} \mathbf{N}^{(2-2\mathbf{H})/q} (q!q)^{-1} V_{\mathbf{N}}$ converges, in $L^2(\Omega)$, to the standard d -parameter Rosenblatt sheet with self-similarity parameter $1 + (2\mathbf{H} - 2)/q$ evaluated at time $\mathbf{1}$, where $c_{1,\mathbf{H}}$ given by (3.9).*

Our proof of Theorem 1.1 follows a strategy introduced by Tudor and Viens in [15], based on the use of chaotic expansion into multiple Wiener-Itô integrals. Let us sketch it. Since the Hermite random field $Z^{q,\mathbf{H}}$ is an element of the q -th Wiener chaos, we can firstly rely on the product formula for multiple integrals to obtain that the quadratic variation $V_{\mathbf{N}}$ can be decomposed into a sum of multiple integrals of even orders from 2 to $2q$. Secondly, by using the isometric property of multiple Wiener-Itô integrals and after

checking the $L^2([0, 1]^d)^2$ convergence of its kernel, we will prove that the projection onto the second Wiener chaos converges in $L^2(\Omega)$ to the d -parameter Rosenblatt random variable. Finally, we will check that all the remaining terms in the chaotic expansion are asymptotically negligible.

In conclusion, it is worth pointing out that, irrespective of the self-similarity parameter, the (properly normalized) quadratic variation of any *non-Gaussian* multiparameter Hermite random fields exhibits a convergence to a random variable belonging to the second Wiener chaos. It is in strong contrast with what happens in the Gaussian case ($q = 1$), where either central or non-central limit theorems may arise, depending on the value of the self-similarity parameter.

The remainder of the paper is structured as follows. Section 2 contains some preliminaries and useful notation. The proof of our main result, namely Theorem 1.1, is then provided in Section 3.

2 Preliminaries

This section describes the notation and the mathematical objects (together with their main properties) that are used throughout this paper.

2.1 Notations

Fix an integer $d \geq 1$. In what follows, we shall systematically use bold notation when dealing with multi-indexed quantities. We thus write $\mathbf{a} = (a_1, a_2, \dots, a_d)$, $\mathbf{ab} = (a_1b_1, a_2b_2, \dots, a_db_d)$ or $\mathbf{a/b} = (a_1/b_1, a_2/b_2, \dots, a_d/b_d)$. Similarly, $[\mathbf{a}, \mathbf{b}] = \prod_{i=1}^d [a_i, b_i]$, $(\mathbf{a}, \mathbf{b}) = \prod_{i=1}^d (a_i, b_i)$. Summation is as follows: $\sum_{\mathbf{i}=1}^{\mathbf{N}} a_{\mathbf{i}} = \sum_{i_1=1}^{N_1} \sum_{i_2=1}^{N_2} \dots \sum_{i_d=1}^{N_d} a_{i_1, i_2, \dots, i_d}$ whereas, for products, we shall write $\mathbf{a}^{\mathbf{b}} = \prod_{i=1}^d a_i^{b_i}$. Finally, we shall write $\mathbf{a} < \mathbf{b}$ (resp. $\mathbf{a} \leq \mathbf{b}$) whenever $a_1 < b_1, a_2 < b_2, \dots, a_d < b_d$ (resp. $a_1 \leq b_1, a_2 \leq b_2, \dots, a_d \leq b_d$).

2.2 Multiple Wiener-Itô integrals

We will now briefly review the theory of multiple Wiener-Itô integrals with respect to Brownian sheet, as described e.g. in Nualart's book [8] (chapter 1 therein) or in [9,

Section 3]. Let $f \in L^2((\mathbb{R}^d)^q)$ and let us denote by $I_q^W(f)$ the q -fold multiple Wiener-Itô integral of f with respect to the standard two-sided Brownian sheet $(W_t)_{t \in \mathbb{R}^d}$. In symbols, such an integral is written

$$I_q^W(f) = \int_{(\mathbb{R}^d)^q} f(\mathbf{u}_1, \dots, \mathbf{u}_q) dW_{\mathbf{u}_1} \dots dW_{\mathbf{u}_q}.$$

Moreover, one has $I_q^W(f) = I_q^W(\tilde{f})$, where \tilde{f} is the symmetrization of f defined by

$$\tilde{f}(\mathbf{u}_1, \dots, \mathbf{u}_q) = \frac{1}{q!} \sum_{\sigma \in \mathfrak{S}_q} f(\mathbf{u}_{\sigma(1)}, \dots, \mathbf{u}_{\sigma(q)}). \quad (2.1)$$

The set of random variables of the form $I_q^W(f)$, when f runs over $L^2((\mathbb{R}^d)^q)$, is called the q th Wiener chaos of W . Furthermore, if $f \in L^2((\mathbb{R}^d)^p)$ and $g \in L^2((\mathbb{R}^d)^q)$ are two symmetric functions, then

$$I_p^W(f) I_q^W(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}^W(f \otimes_r g), \quad (2.2)$$

where the contraction $f \otimes_r g$, which belongs to $L^2((\mathbb{R}^d)^{p+q-2r})$ for every $r = 0, 1, \dots, p \wedge q$, is given by

$$\begin{aligned} f \otimes_r g(\mathbf{u}_1, \dots, \mathbf{u}_{p-r}, \mathbf{v}_1, \dots, \mathbf{v}_{q-r}) \\ = \int_{(\mathbb{R}^d)^r} f(\mathbf{u}_1, \dots, \mathbf{u}_{p-r}, \mathbf{a}_1, \dots, \mathbf{a}_r) g(\mathbf{v}_1, \dots, \mathbf{v}_{q-r}, \mathbf{a}_1, \dots, \mathbf{a}_r) d\mathbf{a}_1 \dots d\mathbf{a}_r. \end{aligned} \quad (2.3)$$

For any $r = 0, \dots, p \wedge q$, Cauchy-Schwarz inequality yields

$$\|f \otimes_r g\|_{L^2((\mathbb{R}^d)^{p+q-2r})} \leq \|f \otimes_r g\|_{L^2((\mathbb{R}^d)^{p+q-2r})} \leq \|f\|_{L^2((\mathbb{R}^d)^p)} \|g\|_{L^2((\mathbb{R}^d)^q)}, \quad (2.4)$$

Also, $f \otimes_p g = \langle f, g \rangle_{L^2((\mathbb{R}^d)^p)}$ when $q = p$. Furthermore, multiple Wiener integrals satisfy the following isometry and orthogonality properties

$$E[I_p^W(f) I_q^W(g)] = \begin{cases} p! \langle \tilde{f}, \tilde{g} \rangle_{L^2((\mathbb{R}^d)^p)} & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

2.3 Multiparameter Hermite Random Fields

Let us now introduce our main object of interest in this paper, the so-called multiparameter Hermite random field. We follow the definition given by Tudor in [14, Chapter 4].

Definition 2.1. Let $q, d \geq 1$ be two integers and let $\mathbf{H} = (H_1, \dots, H_d)$ be a vector belonging to $(\frac{1}{2}, 1)^d$. The d -parameter Hermite random field of order q and self-similarity parameter \mathbf{H} is any random field of the form

$$\begin{aligned} Z^{q, \mathbf{H}}(\mathbf{t}) &= c_{q, \mathbf{H}} \int_{(\mathbb{R}^d)^q} dW_{u_{1,1}, \dots, u_{1,d}} \dots dW_{u_{q,1}, \dots, u_{q,d}} \\ &\quad \times \left(\int_0^{t_1} da_1 \dots \int_0^{t_d} da_d \prod_{j=1}^q (a_1 - u_{j,1})_+^{-(\frac{1}{2} + \frac{1-H_1}{q})} \dots (a_d - u_{j,d})_+^{-(\frac{1}{2} + \frac{1-H_d}{q})} \right) \\ &= c_{q, \mathbf{H}} \int_{(\mathbb{R}^d)^q} dW_{\mathbf{u}_1} \dots dW_{\mathbf{u}_q} \int_0^{\mathbf{t}} d\mathbf{a} \prod_{j=1}^q (\mathbf{a} - \mathbf{u}_j)_+^{-(\frac{1}{2} + \frac{1-\mathbf{H}}{q})}, \end{aligned} \quad (2.5)$$

where $x_+ = \max(x, 0)$, W is a standard two-sided Brownian sheet, and $c(q, \mathbf{H})$ is the positive constant depending only on q and \mathbf{H} chosen so that $E[Z^{q, \mathbf{H}}(\mathbf{1})^2] = 1$.

The above integral (2.5) represents a multiple Wiener-Itô integral of order q with respect to the standard two-sided Brownian sheet W .

In many occasions (like when one wants to simulate $Z^{q, \mathbf{H}}$, or when one looks for constructing a stochastic calculus with respect to it), the following finite-time interval representation for $Z^{q, \mathbf{H}}$ may be interested as well:

$$\begin{aligned} Z^{q, \mathbf{H}}(\mathbf{t}) &\stackrel{(d)}{=} b_{q, \mathbf{H}} \int_0^{t_1} \dots \int_0^{t_d} dW_{u_{1,1}, \dots, u_{1,d}} \dots \int_0^{t_1} \dots \int_0^{t_d} dW_{u_{q,1}, \dots, u_{q,d}} \\ &\quad \times \left(\int_{u_{1,1} \vee \dots \vee u_{q,1}}^{t_1} da_1 \partial_1 K^{H'_1}(a_1, u_{1,1}) \dots \partial_1 K^{H'_1}(a_1, u_{q,1}) \right) \\ &\quad \vdots \\ &\quad \times \left(\int_{u_{1,d} \vee \dots \vee u_{q,d}}^{t_d} da_d \partial_d K^{H'_d}(a_d, u_{1,d}) \dots \partial_d K^{H'_d}(a_d, u_{q,d}) \right) \\ &= b_{q, \mathbf{H}} \int_{[0, \mathbf{t}]^q} dW_{\mathbf{u}_1} \dots dW_{\mathbf{u}_q} \prod_{j=1}^d \int_{u_{1,j} \vee \dots \vee u_{q,j}}^{t_j} da \partial_1 K^{H'_j}(a, u_{1,j}) \dots \partial_1 K^{H'_j}(a, u_{q,j}). \end{aligned} \quad (2.6)$$

In (2.6), K^H stands for the usual kernel appearing in the classical expression of the fractional Brownian motion B^H as a Volterra integral with respect to Brownian motion (see e.g. [6, 7]), that is, $B_t^H = \int_0^t K^H(t, s) dB_s$, whereas

$$b_{q, \mathbf{H}} := \frac{(\mathbf{H}(2\mathbf{H} - 1))^{1/2}}{(q!(\mathbf{H}'(2\mathbf{H}' - 1))^q)^{1/2}} = (\sqrt{q!})^{d-1} \prod_{j=1}^d \frac{(H_j(2H_j - 1))^{1/2}}{(q!(H'_j(2H'_j - 1))^q)^{1/2}} \quad (2.7)$$

is the unique positive constant ensuring that $E[Z^{q,\mathbf{H}}(\mathbf{1})^2] = 1$, where

$$\mathbf{H}' := 1 + \frac{\mathbf{H} - 1}{q} \quad (\iff (2\mathbf{H}' - 2)q = 2\mathbf{H} - 2). \quad (2.8)$$

For a proof of (2.6) when $d = 2$, we refer to Tudor [14, Chapter 4]. Extension to any value of d as presented here is straightforward.

3 Proof of Theorem 1.1

We are now in a position to give the proof of our Theorem 1.1. It is divided into three steps.

3.1 Expanding into Wiener chaos

In preparation for analysing the quadratic variation (1.1), let us find an explicit expression for the chaos decomposition of $V_{\mathbf{N}}$. Using (2.6) and proceeding by induction on the dimension d , we can write $\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q,\mathbf{H}}$ as a q -th Wiener Itô integral with respect to the standard two-sided Brownian sheet $(W_{\mathbf{t}})_{\mathbf{t} \in \mathbb{R}^d}$ as follows: for every $0 \leq i \leq N - 1$, one has

$$\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q,\mathbf{H}} = I_q(f_{i,\mathbf{N}}), \quad (3.1)$$

where

$$f_{i,\mathbf{N}}(\mathbf{x}_1, \dots, \mathbf{x}_q) = b_{q,\mathbf{H}} \prod_{j=1}^d f_{i_j, N_j}(x_{1,j}, \dots, x_{q,j}), \quad (3.2)$$

with $f_{i,N}(x_1, \dots, x_q)$ denoting the expression

$$\begin{aligned} & \mathbb{1}_{[0, \frac{i+1}{N}]}(x_1 \vee \dots \vee x_q) \int_{x_1 \vee \dots \vee x_q}^{\frac{i+1}{N}} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_q) \\ & - \mathbb{1}_{[0, \frac{i}{N}]}(x_1 \vee \dots \vee x_q) \int_{x_1 \vee \dots \vee x_q}^{\frac{i}{N}} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_q), \end{aligned} \quad (3.3)$$

and with $b_{q,\mathbf{H}}$ and \mathbf{H}' given by (2.7) and (2.8) respectively. Indeed, for $d = 1$, see [2, Section 3, p.8], it reduces to

$$\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q,H} = Z_{\frac{i+1}{N}}^{q,H} - Z_{\frac{i}{N}}^{q,H} = b_{q,H} I_q(f_{i,N}),$$

while for $d = 2$, it is easy to verify that

$$\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q, \mathbf{H}} = Z_{\frac{i+1}{N}, \frac{j+1}{M}}^{q, H_1, H_2} - Z_{\frac{i}{N}, \frac{j+1}{M}}^{q, H_1, H_2} - Z_{\frac{i+1}{N}, \frac{j}{M}}^{q, H_1, H_2} + Z_{\frac{i}{N}, \frac{j}{M}}^{q, H_1, H_2} = I_q(f_{i,j,N,M})$$

where

$$\begin{aligned} & f_{i,j,N,M}(x_1, y_1, \dots, x_q, y_q) \\ &= b_{q, H_1, H_2} \mathbb{1}_{[0, \frac{i+1}{N}]}(x_1 \vee \dots \vee x_q) \int_{x_1 \vee \dots \vee x_q}^{\frac{i+1}{N}} du \partial_1 K^{H'_1}(u, x_1) \dots \partial_1 K^{H'_1}(u, x_q) \\ & \quad \times \mathbb{1}_{[0, \frac{j+1}{M}]}(y_1 \vee \dots \vee y_q) \int_{y_1 \vee \dots \vee y_q}^{\frac{j+1}{M}} dv \partial_1 K^{H'_2}(v, y_1) \dots \partial_1 K^{H'_2}(v, y_q) \\ & - b_{q, H_1, H_2} \mathbb{1}_{[0, \frac{i+1}{N}]}(x_1 \vee \dots \vee x_q) \int_{x_1 \vee \dots \vee x_q}^{\frac{i+1}{N}} du \partial_1 K^{H'_1}(u, x_1) \dots \partial_1 K^{H'_1}(u, x_q) \\ & \quad \times \mathbb{1}_{[0, \frac{j}{M}]}(y_1 \vee \dots \vee y_q) \int_{y_1 \vee \dots \vee y_q}^{\frac{j}{M}} dv \partial_1 K^{H'_2}(v, y_1) \dots \partial_1 K^{H'_2}(v, y_q) \\ & - b_{q, H_1, H_2} \mathbb{1}_{[0, \frac{i}{N}]}(x_1 \vee \dots \vee x_q) \int_{x_1 \vee \dots \vee x_q}^{\frac{i}{N}} du \partial_1 K^{H'_1}(u, x_1) \dots \partial_1 K^{H'_1}(u, x_q) \\ & \quad \times \mathbb{1}_{[0, \frac{j+1}{M}]}(y_1 \vee \dots \vee y_q) \int_{y_1 \vee \dots \vee y_q}^{\frac{j+1}{M}} dv \partial_1 K^{H'_2}(v, y_1) \dots \partial_1 K^{H'_2}(v, y_q) \\ & + b_{q, H_1, H_2} \mathbb{1}_{[0, \frac{i}{N}]}(x_1 \vee \dots \vee x_q) \int_{x_1 \vee \dots \vee x_q}^{\frac{i}{N}} du \partial_1 K^{H'_1}(u, x_1) \dots \partial_1 K^{H'_1}(u, x_q) \\ & \quad \times \mathbb{1}_{[0, \frac{j}{M}]}(y_1 \vee \dots \vee y_q) \int_{y_1 \vee \dots \vee y_q}^{\frac{j}{M}} dv \partial_1 K^{H'_2}(v, y_1) \dots \partial_1 K^{H'_2}(v, y_q) \\ & = b_{q, H_1, H_2} f_{i,N}(x_1, \dots, x_q) f_{j,M}(y_1, \dots, y_q). \end{aligned}$$

The last equality above is obtained by grouping each term of $f_{i,j,N,M}$ together. Suppose that the expression (3.1), (3.2) is true for d , that is the kernel of $\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q, \mathbf{H}}$ is equal to

$$\begin{aligned} & b_{q, \mathbf{H}} \sum_{(r_1, \dots, r_d) \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d r_i} \prod_{j=1}^d \mathbb{1}_{[0, \frac{i_j + r_j}{N_j}]}(x_{1,j} \vee \dots \vee x_{q,j}) \\ & \quad \times \int_{x_{1,j} \vee \dots \vee x_{q,j}}^{\frac{i_j + r_j}{N_j}} du \partial_1 K^{H'_j}(u, x_{1,j}) \dots \partial_1 K^{H'_j}(u, x_{q,j}) \\ & = b_{q, \mathbf{H}} \prod_{j=1}^d f_{i_j, N_j}(x_{1,j}, \dots, x_{q,j}). \end{aligned}$$

Then, for the case $d + 1$ we have

$$\begin{aligned}
\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q, \mathbf{H}} &= \sum_{\mathbf{r} \in \{0,1\}^{d+1}} (-1)^{d+1 - \sum_{i=1}^{d+1} r_i} Z_{\frac{i+r}{N}}^{q, \mathbf{H}} \\
&= \sum_{(r_1, \dots, r_d) \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d r_i} Z_{\left(\frac{i_1+r_1}{N_1}, \dots, \frac{i_d+r_d}{N_d}, \frac{i_{d+1}+1}{N_{d+1}}\right)}^{q, \mathbf{H}} \\
&\quad + \sum_{(r_1, \dots, r_d) \in \{0,1\}^d} (-1)^{d+1 - \sum_{i=1}^d r_i} Z_{\left(\frac{i_1+r_1}{N_1}, \dots, \frac{i_d+r_d}{N_d}, \frac{i_{d+1}}{N_{d+1}}\right)}^{q, \mathbf{H}} \\
&= \sum_{(r_1, \dots, r_d) \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d r_i} \left(Z_{\left(\frac{i_1+r_1}{N_1}, \dots, \frac{i_d+r_d}{N_d}, \frac{i_{d+1}+1}{N_{d+1}}\right)}^{q, \mathbf{H}} - Z_{\left(\frac{i_1+r_1}{N_1}, \dots, \frac{i_d+r_d}{N_d}, \frac{i_{d+1}}{N_{d+1}}\right)}^{q, \mathbf{H}} \right)
\end{aligned}$$

It belongs to the q -Wiener chaos with the kernel $f_{\mathbf{i}, \mathbf{N}}$ given by

$$\begin{aligned}
f_{\mathbf{i}, \mathbf{N}} &= b_{q, \mathbf{H}} \sum_{(r_1, \dots, r_d) \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d r_i} \prod_{j=1}^d \mathbb{1}_{[0, \frac{i_j+r_j}{N_j}]}(x_{1,j} \vee \dots \vee x_{q,j}) \\
&\quad \times \int_{x_{1,j} \vee \dots \vee x_{q,j}}^{\frac{i_j+r_j}{N_j}} du \partial_1 K^{H'_j}(u, x_{1,j}) \dots \partial_1 K^{H'_j}(u, x_{q,j}) \\
&\quad \times \left(\int_{x_{1,d+1} \vee \dots \vee x_{q,d+1}}^{\frac{i_{d+1}+1}{N_{d+1}}} du' \partial_1 K^{H'_{d+1}}(u', x_{1,d+1}) \dots \partial_1 K^{H'_{d+1}}(u', x_{q,d+1}) \right. \\
&\quad \left. - \int_{x_{1,d+1} \vee \dots \vee x_{q,d+1}}^{\frac{i_{d+1}}{N_{d+1}}} du' \partial_1 K^{H'_{d+1}}(u', x_{1,d+1}) \dots \partial_1 K^{H'_{d+1}}(u', x_{q,d+1}) \right).
\end{aligned}$$

By the induction hypothesis, one gets $f_{\mathbf{i}, \mathbf{N}} = b_{q, \mathbf{H}} \prod_{j=1}^{d+1} f_{i_j, N_j}(x_{1,j}, \dots, x_{q,j})$, which is our desired expression.

Next, by applying the product formula (2.2), we can write

$$\left(\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q, \mathbf{H}} \right)^2 - E \left[\left(\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q, \mathbf{H}} \right)^2 \right] = \sum_{r=0}^{q-1} r! \binom{q}{r}^2 I_{2q-2r}(f_{\mathbf{i}, \mathbf{N}} \tilde{\otimes}_r f_{\mathbf{i}, \mathbf{N}}). \quad (3.4)$$

Let us compute the contractions appearing in the right-hand side of (3.4). For every $0 \leq r \leq q - 1$, we have

$$\begin{aligned}
&(f_{\mathbf{i}, \mathbf{N}} \otimes_r f_{\mathbf{i}, \mathbf{N}})(\mathbf{x}_1, \dots, \mathbf{x}_{2q-2r}) \\
&= \int_{([0,1]^d)^r} d\mathbf{a}_1 \dots d\mathbf{a}_r f_{\mathbf{i}, \mathbf{N}}(\mathbf{x}_1, \dots, \mathbf{x}_{q-r}, \mathbf{a}_1, \dots, \mathbf{a}_r) \\
&\quad \times f_{\mathbf{i}, \mathbf{N}}(\mathbf{x}_{q-r+1}, \dots, \mathbf{x}_{2q-2r}, \mathbf{a}_1, \dots, \mathbf{a}_r)
\end{aligned}$$

$$\begin{aligned}
&= b_{q,\mathbf{H}}^2 \int_{([0,1]^d)^r} d\mathbf{a}_1 \dots d\mathbf{a}_r \prod_{j=1}^d f_{i_j, N_j}(x_{1,j}, \dots, x_{q-r,j}, a_{1,j}, \dots, a_{r,j}) \\
&\quad \times \prod_{j=1}^d f_{i_j, N_j}(x_{q-r+1,j}, \dots, x_{2q-2r,j}, a_{1,j}, \dots, a_{r,j}) \\
&= b_{q,\mathbf{H}}^2 \prod_{j=1}^d (f_{i_j, N_j} \otimes_r f_{i_j, N_j})(x_{1,j}, \dots, x_{2q-2r,j}). \tag{3.5}
\end{aligned}$$

where

$$\begin{aligned}
&(f_{i,N} \otimes_r f_{i,N})(x_1, \dots, x_{2q-2r}) = (H'(2H' - 1))^r \\
&\times \left\{ \mathbb{1}_{[0, \frac{i+1}{N}]}(x_1 \vee \dots x_{q-r}) \int_{x_1 \vee \dots x_{q-r}}^{\frac{i+1}{N}} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_{q-r}) \right. \\
&\quad \times \mathbb{1}_{[0, \frac{i+1}{N}]}(x_{q-r+1} \vee \dots x_{2q-2r}) \int_{x_{q-r+1} \vee \dots x_{2q-2r}}^{\frac{i+1}{N}} du' \partial_1 K^{H'}(u', x_{q-r+1}) \dots \\
&\quad \quad \quad \dots \partial_1 K^{H'}(u', x_{2q-2r}) |u - u'|^{(2H'-2)r} \\
&- \mathbb{1}_{[0, \frac{i+1}{N}]}(x_1 \vee \dots x_{q-r}) \int_{x_1 \vee \dots x_{q-r}}^{\frac{i+1}{N}} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_{q-r}) \\
&\quad \times \mathbb{1}_{[0, \frac{i}{N}]}(x_{q-r+1} \vee \dots x_{2q-2r}) \int_{x_{q-r+1} \vee \dots x_{2q-2r}}^{\frac{i}{N}} du' \partial_1 K^{H'}(u', x_{q-r+1}) \dots \\
&\quad \quad \quad \dots \partial_1 K^{H'}(u', x_{2q-2r}) |u - u'|^{(2H'-2)r} \\
&- \mathbb{1}_{[0, \frac{i}{N}]}(x_1 \vee \dots x_{q-r}) \int_{x_1 \vee \dots x_{q-r}}^{\frac{i}{N}} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_{q-r}) \\
&\quad \times \mathbb{1}_{[0, \frac{i+1}{N}]}(x_{q-r+1} \vee \dots x_{2q-2r}) \int_{x_{q-r+1} \vee \dots x_{2q-2r}}^{\frac{i+1}{N}} du' \partial_1 K^{H'}(u', x_{q-r+1}) \dots \\
&\quad \quad \quad \dots \partial_1 K^{H'}(u', x_{2q-2r}) |u - u'|^{(2H'-2)r} \\
&+ \mathbb{1}_{[0, \frac{i}{N}]}(x_1 \vee \dots x_{q-r}) \int_{x_1 \vee \dots x_{q-r}}^{\frac{i}{N}} du \partial_1 K^{H'}(u, x_1) \dots \partial_1 K^{H'}(u, x_{q-r}) \\
&\quad \times \mathbb{1}_{[0, \frac{i}{N}]}(x_{q-r+1} \vee \dots x_{2q-2r}) \int_{x_{q-r+1} \vee \dots x_{2q-2r}}^{\frac{i}{N}} du' \partial_1 K^{H'}(u', x_{q-r+1}) \dots \\
&\quad \quad \quad \dots \partial_1 K^{H'}(u', x_{2q-2r}) |u - u'|^{(2H'-2)r} \Big\}. \tag{3.6}
\end{aligned}$$

(See [2, page 10] for a detailed computation of the expression (3.6).) Moreover, since

$Z^{q,\mathbf{H}}$ is \mathbf{H} -self-similar and has stationary increments, one has

$$\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q,\mathbf{H}} \stackrel{(d)}{=} \mathbf{N}^{-\mathbf{H}} \Delta Z_{[i, i+1]}^{q,\mathbf{H}} \stackrel{(d)}{=} \mathbf{N}^{-\mathbf{H}} Z_{[0,1]}^{q,\mathbf{H}}.$$

It follows that

$$E \left[\mathbf{N}^{2\mathbf{H}} \left(\Delta Z_{[\frac{i}{N}, \frac{i+1}{N}]}^{q,\mathbf{H}} \right)^2 \right] = E[Z^{q,\mathbf{H}}(\mathbf{1})^2] = 1.$$

As a consequence, we have

$$V_{\mathbf{N}} = F_{2q,\mathbf{N}} + c_{2q-2} F_{2q-2,\mathbf{N}} + \dots + c_4 F_{4,\mathbf{N}} + c_2 F_{2,\mathbf{N}}. \quad (3.7)$$

where $c_{2q-2r} = r! \binom{q}{r}^2$, $r = 0, \dots, q-1$, are the combinational constants coming from the product formula, and

$$F_{2q-2r,\mathbf{N}} := \mathbf{N}^{2\mathbf{H}-1} I_{2q-2r} \left(\sum_{i=0}^{\mathbf{N}-1} f_{i,\mathbf{N}} \widetilde{\otimes}_r f_{i,\mathbf{N}} \right), \quad (3.8)$$

for the kernels $f_{i,\mathbf{N}} \widetilde{\otimes}_r f_{i,\mathbf{N}}$ computed in (3.5).

3.2 Evaluating the $L^2(\Omega)$ -norm

Set

$$\begin{aligned} c_{1,\mathbf{H}} &= \frac{2! 2^d b_{q,\mathbf{H}}^4 (\mathbf{H}'(2\mathbf{H}' - 1))^{2q}}{(4\mathbf{H}' - 3)(4\mathbf{H}' - 2)[(2\mathbf{H}' - 2)(q - 1) + 1]^2[(\mathbf{H}' - 1)(q - 1) + 1]^2} \\ &= \frac{(q!)^{2(d-1)}}{2^{d-1}} \prod_{j=1}^d \frac{4(H_j(2H_j - 1))^2}{(q!)^2(4H_j' - 3)(4H_j' - 2)[(2H_j' - 2)(q - 1) + 1]^2[(H_j' - 1)(q - 1) + 1]^2}. \end{aligned} \quad (3.9)$$

We claim that

$$\lim_{\mathbf{N} \rightarrow \infty} E[c_{1,\mathbf{H}}^{-1} \mathbf{N}^{2(2-2\mathbf{H}')} c_2^{-2} V_{\mathbf{N}}^2] = 1. \quad (3.10)$$

Let us prove (3.10). Due to the orthogonality property for Wiener chaos of different orders, it is sufficient to evaluate the $L^2(\Omega)$ -norm of each multiple Wiener-Itô integrals appearing in the chaotic decomposition (3.7) of $V_{\mathbf{N}}$. Let us start with the double integral:

$$F_{2,\mathbf{N}} = \mathbf{N}^{2\mathbf{H}-1} I_2 \left(\sum_{i=0}^{\mathbf{N}-1} f_{i,\mathbf{N}} \otimes_{q-1} f_{i,\mathbf{N}} \right).$$

Since the kernel $\sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}}$ is symmetric, one has

$$\begin{aligned} E[F_{2,\mathbf{N}}^2] &= 2N^{4\mathbf{H}-2} \left\| \sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}} \right\|_{L^2([0,1]^d)^2}^2 \\ &= 2N^{4\mathbf{H}-2} \sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} \langle f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}}, f_{\mathbf{k},\mathbf{N}} \otimes_{q-1} f_{\mathbf{k},\mathbf{N}} \rangle_{L^2([0,1]^d)^2}. \end{aligned}$$

Let us now compute the scalar products in the above expression. By using (3.5), one gets

$$\begin{aligned} &\langle f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}}, f_{\mathbf{k},\mathbf{N}} \otimes_{q-1} f_{\mathbf{k},\mathbf{N}} \rangle_{L^2([0,1]^d)^2} \\ &= b_{q,\mathbf{H}}^4 \prod_{j=1}^d \langle f_{i_j, N_j} \otimes_{q-1} f_{i_j, N_j}, f_{k_j, N_j} \otimes_{q-1} f_{k_j, N_j} \rangle_{L^2([0,1]^d)} \\ &= b_{q,\mathbf{H}}^4 (\mathbf{H}'(2\mathbf{H}' - 1))^{2q} \prod_{j=1}^d \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} du_j \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} dv_j \int_{\frac{k_j}{N_j}}^{\frac{k_j+1}{N_j}} du'_j \int_{\frac{k_j}{N_j}}^{\frac{k_j+1}{N_j}} dv'_j \\ &\quad \times |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)} \\ &\quad \times |u_j - u'_j|^{2H'_j-2} |v_j - v'_j|^{2H'_j-2}. \end{aligned}$$

The change of variables $u' = (u - \frac{i}{N})N$ for each u_j, u'_j, v_j, v'_j with j from 1 to d yields

$$\begin{aligned} E[F_{2,\mathbf{N}}^2] &= 2b_{q,\mathbf{H}}^4 (\mathbf{H}'(2\mathbf{H}' - 1))^{2q} N^{4\mathbf{H}-2} N^{-4} N^{-(2\mathbf{H}'-2)2q} \\ &\quad \times \sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} \prod_{j=1}^d \int_0^1 du_j \int_0^1 dv_j \int_0^1 du'_j \int_0^1 dv'_j |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)} \\ &\quad \times |u_j - u'_j + i_j - k_j|^{2H'_j-2} |v_j - v'_j + i_j - k_j|^{2H'_j-2}. \end{aligned}$$

We split the sum $\sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1}$ appearing in $E[F_{2,\mathbf{N}}^2]$ just above into

$$\sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} = \sum_{\substack{\mathbf{i},\mathbf{k}=0 \\ \exists 1 \leq j \leq d: i_j = k_j}}^{\mathbf{N}-1} + \sum_{\substack{\mathbf{i},\mathbf{k}=0 \\ \forall j: i_j \neq k_j}}^{\mathbf{N}-1}.$$

For the first term, without loss of generality, we can assume that $i_1 = k_1$ and $i_j \neq k_j$ for all $j \neq 1$. Then,

$$\begin{aligned}
& \mathbf{N}^{-2} \sum_{\substack{\mathbf{i}, \mathbf{k}=0 \\ i_1=k_1}}^{\mathbf{N}-1} \prod_{j=1}^d \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)} \\
& \quad \times |u_j - u'_j + i_j - k_j|^{2H'_j-2} |v_j - v'_j + i_j - k_j|^{2H'_j-2} \\
& = N_1^{-1} \int_{[0,1]^4} du_1 dv_1 du'_1 dv'_1 |u_1 - v_1|^{(2H'_1-2)(q-1)} |u'_1 - v'_1|^{(2H'_1-2)(q-1)} \\
& \quad \times |u_1 - u'_1|^{2H'_1-2} |v_1 - v'_1|^{2H'_1-2} \\
& \quad \times \prod_{j=2}^d \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)} \\
& \quad \times 2N_j^{-2} \sum_{\substack{i_j, k_j=0 \\ i_j > k_j}}^{N_j-1} |u_j - u'_j + i_j - k_j|^{2H'_j-2} |v_j - v'_j + i_j - k_j|^{2H'_j-2}.
\end{aligned}$$

Following [2] or [15], one has that

$$\begin{aligned}
& N^{-2} \sum_{\substack{i, k=0 \\ i > k}}^{N-1} |u - u' + i - k|^{2H'-2} |v - v' + i - k|^{2H'-2} \\
& = N^{2(2H'-2)} \frac{1}{N} \sum_{n=1}^N \left(1 - \frac{n}{N}\right) \left| \frac{u - u'}{N} + \frac{n}{N} \right|^{2H'-2} \left| \frac{v - v'}{N} + \frac{n}{N} \right|^{2H'-2}
\end{aligned}$$

is asymptotically equivalent to

$$N^{2(2H'-2)} \int_0^1 (1-x) x^{4H'-4} dx = N^{2(2H'-2)} \frac{1}{(4H'-3)(4H'-2)}.$$

It follows that

$$\begin{aligned}
& \mathbf{N}^{-2} \sum_{\substack{\mathbf{i}, \mathbf{k}=0 \\ i_1=k_1}}^{\mathbf{N}-1} \prod_{j=1}^d \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)} \\
& \quad \times |u_j - u'_j + i_j - k_j|^{2H'_j-2} |v_j - v'_j + i_j - k_j|^{2H'_j-2} \\
& \approx N_1^{-1} \int_{[0,1]^4} du_1 dv_1 du'_1 dv'_1 |u_1 - v_1|^{(2H'_1-2)(q-1)} |u'_1 - v'_1|^{(2H'_1-2)(q-1)} \\
& \quad \times |u_1 - u'_1|^{2H'_1-2} |v_1 - v'_1|^{2H'_1-2} \\
& \quad \times \prod_{j=2}^d 2N_j^{2(2H'_j-2)} \frac{1}{(4H'_j-3)(4H'_j-2)[(2H'_j-2)(q-1)+1]^2[(H'_j-1)(q-1)+1]^2}.
\end{aligned}$$

Similarly for the second term, where $i_j \neq k_j$ for all $1 \leq j \leq d$, we have

$$\begin{aligned} & \mathbf{N}^{-2} \sum_{\substack{\mathbf{i}, \mathbf{k}=0 \\ i_j \neq k_j, \forall j}}^{\mathbf{N}-1} \prod_{j=1}^d \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)(q-1)} |u'_j - v'_j|^{(2H'_j-2)(q-1)} \\ & \quad \times |u_j - u'_j + i_j - k_j|^{2H'_j-2} |v_j - v'_j + i_j - k_j|^{2H'_j-2} \\ & \approx \prod_{j=1}^d 2N_j^{2(2H'_j-2)} \frac{1}{(4H'_j-3)(4H'_j-2)[(2H'_j-2)(q-1)+1]^2[(H'_j-1)(q-1)+1]^2}. \end{aligned}$$

Putting everything together, we conclude that

$$\lim_{\mathbf{N} \rightarrow \infty} E[c_{1,\mathbf{H}}^{-1} \mathbf{N}^{2(2-2\mathbf{H}')} F_{2,\mathbf{N}}^2] = 1. \quad (3.11)$$

Let us now consider the remaining terms $F_{4,\mathbf{N}}, \dots, F_{2q,\mathbf{N}}$ in the chaos decomposition (3.7). Using that $\|\tilde{g}\|_{L^2} \leq \|g\|_{L^2}$ for any square integrable function g , one can write, for every $0 \leq r \leq q-2$,

$$\begin{aligned} E[F_{2q-2r,\mathbf{N}}^2] &= \mathbf{N}^{4\mathbf{H}-2} (2q-2r)! \left\| \sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \tilde{\otimes}_r f_{\mathbf{i},\mathbf{N}} \right\|_{L^2([0,1]^d)^{2q-2r}}^2 \\ &\leq \mathbf{N}^{4\mathbf{H}-2} (2q-2r)! \left\| \sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_r f_{\mathbf{i},\mathbf{N}} \right\|_{L^2([0,1]^d)^{2q-2r}}^2 \\ &= (2q-2r)! \mathbf{N}^{4\mathbf{H}-2} \sum_{\mathbf{i}, \mathbf{k}=0}^{\mathbf{N}-1} \langle f_{\mathbf{i},\mathbf{N}} \otimes_r f_{\mathbf{i},\mathbf{N}}, f_{\mathbf{k},\mathbf{N}} \otimes_r f_{\mathbf{k},\mathbf{N}} \rangle_{L^2([0,1]^d)^{2q-2r}}. \end{aligned}$$

Proceeding as above, we obtain

$$\begin{aligned} & \langle f_{\mathbf{i},\mathbf{N}} \otimes_r f_{\mathbf{i},\mathbf{N}}, f_{\mathbf{k},\mathbf{N}} \otimes_r f_{\mathbf{k},\mathbf{N}} \rangle_{L^2([0,1]^d)^{2q-2r}} \\ &= b_{q,\mathbf{H}}^4 (\mathbf{H}'(2\mathbf{H}'-1))^{2q} \prod_{j=1}^d \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} du_j \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} dv_j \int_{\frac{k_j}{N_j}}^{\frac{k_j+1}{N_j}} du'_j \int_{\frac{k_j}{N_j}}^{\frac{k_j+1}{N_j}} dv'_j \\ & \quad \times |u_j - v_j|^{(2H'_j-2)r} |u'_j - v'_j|^{(2H'_j-2)r} \\ & \quad \times |u_j - u'_j|^{(2H'_j-2)(q-r)} |v_j - v'_j|^{(2H'_j-2)(q-r)}. \end{aligned}$$

Using the change of variables $u' = (u - \frac{i}{N})N$ for each u_j, u'_j, v_j, v'_j with $j = 1, \dots, d$, one obtains

$$\begin{aligned}
E[F_{2q-2r,\mathbf{N}}^2] &\leq (2q-2r)!b_{q,\mathbf{H}}^4(\mathbf{H}'(2\mathbf{H}'-1))^{2q}\mathbf{N}^{4\mathbf{H}-2}\mathbf{N}^{-4}\mathbf{N}^{-(2\mathbf{H}'-2)2q} \\
&\times \sum_{\mathbf{i},\mathbf{k}=0}^{\mathbf{N}-1} \prod_{j=1}^d \int_0^1 du_j \int_0^1 dv_j \int_0^1 du'_j \int_0^1 dv'_j |u_j - v_j|^{(2H'_j-2)r} |u'_j - v'_j|^{(2H'_j-2)r} \\
&\quad \times |u_j - u'_j + i_j - k_j|^{(2H'_j-2)(q-r)} |v_j - v'_j + i_j - k_j|^{(2H'_j-2)(q-r)} \\
&= (2q-2r)!b_{q,\mathbf{H}}^4(\mathbf{H}'(2\mathbf{H}'-1))^{2q}\mathbf{N}^{-2} \\
&\quad \times \prod_{j=1}^d \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)r} |u'_j - v'_j|^{(2H'_j-2)r} \\
&\quad \times \sum_{i_j,k_j=0}^{N_j-1} |u_j - u'_j + i_j - k_j|^{(2H'_j-2)(q-r)} |v_j - v'_j + i_j - k_j|^{(2H'_j-2)(q-r)} \\
&= (2q-2r)!b_{q,\mathbf{H}}^4(\mathbf{H}'(2\mathbf{H}'-1))^{2q}\mathbf{N}^{2(2\mathbf{H}'-2)(q-r)} \\
&\quad \times \prod_{j=1}^d \int_{[0,1]^4} du_j dv_j du'_j dv'_j |u_j - v_j|^{(2H'_j-2)r} |u'_j - v'_j|^{(2H'_j-2)r} \\
&\quad \times \frac{2}{N_j} \sum_{n_j=1}^{N_j} \left(1 - \frac{n_j}{N_j}\right) \left| \frac{u_j - u'_j}{N_j} + \frac{n_j}{N_j} \right|^{(2H'_j-2)(q-r)} \left| \frac{v_j - v'_j}{N_j} + \frac{n_j}{N_j} \right|^{(2H'_j-2)(q-r)}.
\end{aligned}$$

Apart from the diagonal terms, the term $\frac{u-u'}{N}$ is dominated by $\frac{n}{N}$ for n, N large enough.

Using a Riemann sum approximation, one has for $0 \leq r \leq q-2$ and as $\mathbf{N} \rightarrow \infty$,

$$E[\mathbf{N}^{(2-2\mathbf{H}')(2q-2r)} F_{2q-2r,\mathbf{N}}^2] = O(1). \quad (3.12)$$

We deduce that

$$\lim_{\mathbf{N} \rightarrow \infty} E[\mathbf{N}^{2(2-2\mathbf{H}')} F_{2q-2r,\mathbf{N}}^2] = 0, \quad (3.13)$$

and the proof of (3.10) follows.

3.3 Concluding the proof of Theorem 1.1

Thanks to (3.10), in order to understand the asymptotic behavior of the renormalized sequence of $V_{\mathbf{N}}$, it is enough to analyse the convergence of the term

$$I_2 \left(\mathbf{N}^{2\mathbf{H}-1} \mathbf{N}^{2-2\mathbf{H}'} \sum_{\mathbf{i}=0}^{\mathbf{N}-1} f_{\mathbf{i},\mathbf{N}} \otimes_{q-1} f_{\mathbf{i},\mathbf{N}} \right) \quad (3.14)$$

with

$$\begin{aligned}
f_{\mathbf{i}, \mathbf{N}} \otimes_{q-1} f_{\mathbf{i}, \mathbf{N}}(\mathbf{x}_1, \mathbf{x}_2) &= b_{q, \mathbf{H}}^2 \prod_{j=1}^d (f_{i_j, N_j} \otimes_{q-1} f_{i_j, N_j})(x_{1,j}, x_{2,j}) \\
&= b_{q, \mathbf{H}}^2 (\mathbf{H}'(2\mathbf{H}' - 1))^{q-1} \\
&\quad \times \prod_{j=1}^d \left(\mathbb{1}_{[0, \frac{i_j}{N_j}]}(x_{1,j}) \mathbb{1}_{[0, \frac{i_j}{N_j}]}(x_{2,j}) \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} du \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} du' \partial_1 K^{H'_j}(u, x_{1,j}) \right. \\
&\quad \times \partial_1 K^{H'_j}(u', x_{2,j}) |u - u'|^{(2H'_j-2)(q-1)} \\
&\quad + \mathbb{1}_{[0, \frac{i_j}{N_j}]}(x_{1,j}) \mathbb{1}_{[\frac{i_j}{N_j}, \frac{i_j+1}{N_j}]}(x_{2,j}) \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} du \int_{x_{2,j}}^{\frac{i_j+1}{N_j}} du' \partial_1 K^{H'_j}(u, x_{1,j}) \\
&\quad \times \partial_1 K^{H'_j}(u', x_{2,j}) |u - u'|^{(2H'_j-2)(q-1)} \\
&\quad + \mathbb{1}_{[\frac{i_j}{N_j}, \frac{i_j+1}{N_j}]}(x_{1,j}) \mathbb{1}_{[0, \frac{i_j+1}{N_j}]}(x_{2,j}) \int_{x_{1,j}}^{\frac{i_j+1}{N_j}} du \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} du' \partial_1 K^{H'_j}(u, x_{1,j}) \\
&\quad \times \partial_1 K^{H'_j}(u', x_{2,j}) |u - u'|^{(2H'_j-2)(q-1)} \\
&\quad \left. + \mathbb{1}_{[\frac{i_j}{N_j}, \frac{i_j+1}{N_j}]}(x_{1,j}) \mathbb{1}_{[\frac{i_j}{N_j}, \frac{i_j+1}{N_j}]}(x_{2,j}) \int_{x_{1,j}}^{\frac{i_j+1}{N_j}} du \int_{x_{2,j}}^{\frac{i_j+1}{N_j}} du' \partial_1 K^{H'_j}(u, x_{1,j}) \right. \\
&\quad \left. \times \partial_1 K^{H'_j}(u', x_{2,j}) |u - u'|^{(2H'_j-2)(q-1)} \right).
\end{aligned}$$

Among the four terms of each factor in the right-hand side of the above expression, only the first one is not asymptotically negligible in $L^2(\Omega)$, see [2, page 14 and 15] or follow the lines of [15] for details. Furthermore, by the isometry property for multiple Wiener-Itô integrals, evaluating the $L^2(\Omega)$ -limit of a sequence belonging to the second Wiener chaos is equivalent to evaluating the $L^2([0, 1]^d)^2$ -limit of the sequence of their corresponding symmetric kernels. Therefore, we are left to find the limit of $f_2^{\mathbf{N}}$ in $L^2([0, 1]^d)^2$, where

$$\begin{aligned}
f_2^{\mathbf{N}}(\mathbf{x}_1, \mathbf{x}_2) &:= (q!)^{d-1} \prod_{j=1}^d \frac{H_j(2H_j - 1)}{q!(H'_j(2H'_j - 1))^q} (H'_j(2H'_j - 1))^{q-1} N_j^{2H_j-1} N_j^{2-2H'_j} \\
&\quad \times \sum_{i_j=0}^{N_j-1} \mathbb{1}_{[0, \frac{i_j}{N_j}]}(x_{1,j}) \mathbb{1}_{[0, \frac{i_j}{N_j}]}(x_{2,j}) \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} du \int_{\frac{i_j}{N_j}}^{\frac{i_j+1}{N_j}} du' \partial_1 K^{H'_j}(u, x_{1,j}) \\
&\quad \times \partial_1 K^{H'_j}(u', x_{2,j}) |u - u'|^{(2H'_j-2)(q-1)}.
\end{aligned}$$

According to [2, Theorem 3.2], by using integral approximation and Lebesgue dominated convergence theorem, each term under the product for j from 1 to d converges in $L^2([0, 1]^2)$ to the constant a_j times the kernel of a standard Rosenblatt random variable $Z^{2, 2H'_j-1}(1)$ as $N_j \rightarrow \infty$ with

$$a_j = \frac{2H_j(2H_j - 1)}{q!(4H'_j - 3)^{1/2}(4H'_j - 2)^{1/2}[(2H'_j - 2)(q - 1) + 1][(H'_j - 1)(q - 1) + 1]}.$$

Therefore, $f_2^{\mathbf{N}}$ converges in $L^2([0, 1]^d)$ to the constant $2^{\frac{d-1}{2}} c_{1, \mathbf{H}}^{1/2}$ times the kernel of a standard Rosenblatt sheet at time $\mathbf{1}$ as $\mathbf{N} \rightarrow \infty$, which leads to our desired conclusion.

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